Large Deflection Analysis of Uniformly Loaded Annular Membranes

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An iterative technique is employed to obtain approximate solutions to Föppl's nonlinear membrane equations. Four uniformly loaded rotationally symmetric membranes are examined: A) an annulus fixed at both edges, B) an annulus fixed at the outer edge with a rigid plug in the interior, C) an annulus fixed at the outer edge and free of tractions at the inner edge, and D) a solid circular membrane fixed at the outer edge. Numerical results are presented for each problem. The results of case D are compared with the existing exact power series solution presented by Hencky. Stresses and deflections computed by the iterative technique for case D are within 1.14% of those predicted by Hencky.

Nomenclature

a,b = inner and outer radii of annular membranes, respectively
e = dilatation $\epsilon_r + \epsilon_\theta$
E = modulus of elasticity
F = stress function
h = thickness of membrane
N_r, N_θ = in-plane stress resultants in the radial and circumferential directions, respectively
q = normal load intensity
r = radial coordinate
S_r, S_θ = dimensionless stress resultants; $S_r = N_r/(q^2b^2hE)^{1/3}$, $S_\theta = N_\theta/(q^2b^2hE)^{1/3}$
u = in-plane displacement
U = dimensionless displacement $[u/(q^2b^5/h^2E^2)^{1/3}]$
w = transverse displacement
W = dimensionless deflection $[w/(qb^4/Eh)^{1/3}]$
α_1,α_2 = characteristic unknowns
γ = reduced dilatation $[e/(qb/hE)^{2/3}]$
γ_r = reduced radial strain $[\epsilon_r/(qb/hE)^{2/3}]$
γ_θ = reduced circumferential strain $[\epsilon_\theta/(qb/hE)^{2/3}]$
δ = constant
$\epsilon_r, \epsilon_\theta$ = radial and circumferential strains
λ = inner to outer radius ratio a/b
ν = Poisson's ratio
ρ = dimensionless radial coordinate r/b

Introduction

THE anticipated use of extremely thin structural elements for space applications has provided an impetus for the engineer to investigate large deflection phenomena.[1, 2] Such considerations usually give rise to nonlinear differential equations. The complexities of these expressions present serious analytical difficulties when applied to the simplest boundary-value problem.

In particular, the deformation of initially flat membranes, described by Föppl's large deflection theory,[3, 4] is characterized by two coupled nonlinear, partial differential equations. The dimensionless form of these equations, in rotationally symmetric polar coordinates, are

$$(1/\rho)(d/d\rho)[(dF/d\rho)(dW/d\rho)] = -1 \tag{1}$$

and

$$\nabla^4 F = -(1/2\rho)(d/d\rho)[(dW/d\rho)^2] \tag{2}$$

where Eq. (1) originates from transverse equilibrium considerations, and Eq. (2) is obtained from the compatibility of the in-plane strains.

An iterative technique[5] is employed herein to obtain approximate solutions to Föppl's equations. Essentially, the procedure is to neglect the right-hand side of the compatibility equation for the in-plane strains, Eq. (2). The result is an uncoupled system of ordinary differential equations. The solution to these equations is then employed to obtain a more refined approximate solution to Föppl's equations. The resulting approximate solution satisfies the usual equilibrium equations but violates the compatibility of the membrane strains. Successive applications of the technique outlined is capable of providing an accurate solution as is demonstrated in one of the examples presented.

Four problems involving uniformly loaded rotationally symmetric membranes are examined: A) an annulus fixed at both edges, B) an annulus fixed at the outer edge with a rigid plug in the interior, C) an annulus fixed at the outer edge and free of tractions at the inner edge, and D) a solid circular membrane fixed at the outer edge.

Dimensionless Field Equations

In the absence of body forces the nonvanishing in-plane equilibrium equation is

$$(d/d\rho)(\rho S_r) - S_\theta = 0 \tag{3a}$$

or, in terms of displacements,

$$\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}(\rho U)\right] = -\frac{(1-\nu)}{2\rho}\left[\frac{dW}{d\rho}\right]^2 - \frac{1}{2}\frac{d}{d\rho}\left[\left(\frac{dW}{d\rho}\right)^2\right] \tag{3b}$$

where

$$S_r = \frac{1}{\rho}\frac{dF}{d\rho} \qquad S_\theta = \frac{d^2F}{d\rho^2} \qquad S_{r\theta} = 0 \tag{3c}$$

The resultant stress-strain and strain-displacement relations are

$$\gamma_r = S_r - \nu S_\theta = (dU/d\rho) + \tfrac{1}{2}(dW/d\rho)^2 \tag{3d}$$

$$\gamma_\theta = S_\theta - \nu S_r = U/\rho \tag{3e}$$

and the dilatation is

$$\gamma = \gamma_r + \gamma_\theta = \frac{1}{\rho}\frac{d}{d\rho}(\rho U) + \frac{1}{2}\left[\frac{dW}{d\rho}\right]^2 = (1-\nu)(S_r + S_\theta) = (1-\nu)\nabla^2 F \tag{3f}$$

Equation (1) can be recast into a more convenient form

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by putting (3c) in (1) and integrating to obtain

$$\rho S_r(dW/d\rho) = \tfrac{1}{2}[A_1 - \rho^2] \tag{4}$$

where A_1 is a constant of integration.

Statement of the Problems

Approximate solutions of (4) and (2), subject to the four sets of boundary conditions, are sought.

Case A: An annulus fixed at both edges

$$U = W = 0 \qquad \text{at } \rho = 1 \text{ and } \rho = \lambda \tag{5a}$$

Case B: An annulus fixed at the outer edge with a rigid plug in the interior

$$U = W = 0 \qquad \text{at } \rho = 1$$

$$U = S_r(dW/d\rho) = 0 \qquad \text{at } \rho = \lambda \tag{5b}$$

Case C: An annulus fixed at the outer edge and free of tractions at the inner edge

$$U = W = 0 \qquad \text{at } \rho = 1$$

$$S_r = S_r(dW/d\rho) = 0 \qquad \text{at } \rho = \lambda \tag{5c}$$

Case D: A solid circular membrane fixed at the outer edge

$$U = W = 0 \qquad \text{at } \rho = 1$$

$$U = (dW/d\rho) = 0 \qquad \text{at } \rho = 0 \tag{5d}$$

To this end, the right-hand side of (2) is set equal to zero, and Lame's[6] solution is taken for the first approximation of the stress function

$$F_1 = \frac{\alpha_1}{4}\left(\frac{2}{1-\nu}\right)^{1/3}\left[\frac{\rho^2}{2} - \delta\lambda^2 \ln\rho\right] \tag{6}$$

The constant δ must be specified, whereas α_1 is a characteristic unknown of the problem and will be evaluated from the boundary conditions. Thus, $[\rho^2/2 - \delta\lambda^2 \ln\rho]$ describes the functional form of the approximate stress field, and α_1 controls the magnitude of these stresses.

Approaching the problem in this manner violates compatibility of the in-plane strains. Consequently, the relationship between the stress resultant and the applied load cannot be obtained by direct application of the stress-displacement equations. This becomes clear when it is noted that the stresses issuing from (6) represent the stress field associated with a two-dimensional problem of plane elasticity. Thus, (6) does not include the influence of the normal deflection explicitly. However, the stress field is related to the applied load and thereby to the deflection of the membrane by virtue of the characteristic unknown α_1.

Uniformly Loaded Annular Membrane

The first approximation for the radial membrane stress in a uniformly loaded annular membrane is, from (6) and (3c),

$$S_r^{\mathrm{I}} = \frac{\alpha_1}{4}\left(\frac{2}{1-\nu}\right)^{1/3}\left[1 - \frac{\delta\lambda^2}{\rho^2}\right] \tag{7}$$

where δ remains to be specified. For problems requiring a stress free boundary, such as (5c), or finite stresses at the origin, such as (5d), it is seen that δ must equal one or zero, respectively. However, for problems involving in-plane displacement boundary conditions, such as (5a) and (5b), there is no a priori knowledge governing the value of δ. Consequently, since (3e) yields a direct relation between U and the approximate stress (7), substitute (7) into (3e), and stipulate the vanishing of U at either $\rho = 1$ or λ. Then δ is chosen such that the limit of U as λ approaches zero will satisfy the condition that U vanish at the origin. Therefore, from the stipulation at $\rho = \lambda$, one has

$$\delta = -(1-\nu)/(1+\nu) \tag{8}$$

Thus, with δ specified, there is an approximate stress field, for each case, which satisfies one of the radial stress or displacement boundary conditions. The remaining condition will be accounted for by the characteristic unknown α_1. Having established the functional form of the in-plane stresses, W can now be determined.

Substituting (7) into (4), integrating, and making use of the boundary condition at $\rho = 1$ yields, as a first approximation to the deflection,

$$W_{\mathrm{I}} = \left(\frac{1-\nu}{2}\right)^{1/3}\frac{1}{\alpha_1}\left[1 - \rho^2 + (A_1 - \delta\lambda^2)\ln\left(\frac{\rho^2 - \delta\lambda^2}{1 - \delta\lambda^2}\right)\right] \tag{9}$$

where A_1 is determined from the boundary condition at the inner edge.

In order to find the unknown constant α_1, used must be the dilatation equation. Therefore, putting (7) and (9) into (3f), a linear first-order, ordinary differential equation for U is obtained, which is associated with the first approximations to the deflection and the membrane stresses. After integrating the resulting equation and recalling that $U = 0$ at $\rho = 1$, it is found that

$$\rho U = -(1-\nu)\frac{\alpha_1}{4}\left(\frac{2}{1-\nu}\right)^{1/3}(1-\rho^2) + 2\left(\frac{1-\nu}{2}\right)^{2/3}\frac{1}{\alpha_1^2}\int_\rho^1 \rho^3\left(\frac{A_1-\rho^2}{\rho^2-\delta\lambda^2}\right)^2 d\rho \tag{10}$$

The characteristic unknown α_1 is now evaluated from the conditions at $\rho = \lambda$. In the case where U vanishes at $\rho = \lambda$, α_1 follows directly from (10). If, however, the conditions at $\rho = \lambda$ are specified on the radial stress, the stress associated with (10) must be found in order to evaluate α_1. For such cases, after putting (9) and (10) into (3d) and (3e), one has

$$S_r = \frac{\alpha_1}{4}\left(\frac{2}{1-\nu}\right)^{1/3}\left[1 + \frac{1-\nu}{1+\nu}\frac{1}{\rho^2}\right] - \frac{2}{(1+\nu)}\left(\frac{1-\nu}{2}\right)^{2/3}\frac{1}{\alpha_1^2\rho^2}\int_\rho^1 \rho^3\left(\frac{A_1-\rho^2}{\rho^2-\delta\lambda^2}\right)^2 d\rho \tag{11}$$

Hence, for problems such as cases A and B, where U vanishes at $\rho = \lambda$, one has

$$\alpha_1^3 = \frac{4}{(1-\lambda^2)}\int_\lambda^1 \rho^3\left(\frac{A_1-\rho^2}{\rho^2-\delta\lambda^2}\right)^2 d\rho \tag{12}$$

and, for problems such as case C, where S_r vanishes at $\rho = \lambda$, one has

$$\alpha_1^3 = \frac{4(1-\nu)}{[(1-\nu)+(1+\nu)\lambda^2]}\int_\lambda^1 \rho^3\left(\frac{A_1-\rho^2}{\rho^2-\delta\lambda^2}\right)^2 d\rho \tag{13}$$

The unknown α_1 is now given in terms of (12) or (13), and the problems are completely determined (except for the constant A_1). The solutions are characterized by (7), (9), and either (12) or (13), with the appropriate description of δ. These equations represent a first approximation to the solution of Föppl's equations.

It should be noted that the displacement and stresses, (10) and (11), found from (3f) violate the equilibrium equations (3a) and (3b). This was anticipated since (3f) violates the compatibility equation (2). Thus, the relations (10) and (11) are only used to find the characteristic unknown α_1, which in turn relates the lateral loads to the in-plane stress and displacement quantities.

A more refined first approximation for S_r, however, can be found from statical considerations. That is, making use of W_{I} in (3b) yields a second-order ordinary differential equation for an in-plane displacement U_{IR}. The displacement and the corresponding stress resultants obtained now include the effect of the transverse displacement explicitly, whereas the stress resultants defined by (7) are only influenced by the transverse displacement implicitly, through the characteristic unknown α_1. Thus, performing the indicated integrations and making use of the common boundary condition $U = 0$ at $\rho = 1$, one obtains

$$U_{IR} = \frac{2}{\alpha_1^2}\left(\frac{1-\nu}{2}\right)^{2/3}\left[\frac{1}{\rho}\int_\rho^1 \rho G(\rho)d\rho - \left(\rho - \frac{1}{\rho}\right)\frac{A_2}{2}\right] \quad (14)$$

where $G(\rho)$ is defined as

$$G(\rho) = \rho^2\left(\frac{A_1 - \rho^2}{\rho^2 - \delta\lambda^2}\right)^2 + (1-\nu)\int \rho\left(\frac{A_1-\rho^2}{\rho^2-\delta\lambda^2}\right)^2 d\rho \quad (15)$$

and A_2 is a constant, to be determined from the conditions at $\rho = \lambda$. The radial stress associated with (9) and (14) is then found, from (3d) and (3e), to be

$$S_r^{IR} = -\frac{1}{4\alpha_1^2}\left(\frac{1-\nu}{2}\right)^{2/3}\left[\left\{3 - \nu + 2(A_1 - \delta\lambda^2)(A_1 - 3\delta\lambda^2)(1+\nu)\ln\left(\frac{1-\delta\lambda^2}{\rho^2 - \delta\lambda^2}\right) - 4(A_1 - \delta\lambda^2)(1-\nu)\ln(1-\delta\lambda^2) + 4(A_1-\delta\lambda^2)\times \left(\frac{\delta\lambda^2(A_1-\delta\lambda^2)}{(1-\delta\lambda^2)(\rho^2-\delta\lambda^2)} - (1+\nu)\right) + 4A_2\right\}\frac{1}{(1+\nu)\rho^2} - \frac{4(A_1-\delta\lambda^2)}{(1+\nu)}\left(\frac{\delta\lambda^2(A_1-\delta\lambda^2)}{(1-\delta\lambda^2)(\rho^2-\delta\lambda^2)} - (1+\nu)\right) + \frac{4A_2}{1-\nu} - 4(A_1-\delta\lambda^2)\ln(\rho^2-\delta\lambda^2) - \frac{4(A_1-\delta\lambda^2)^2}{(\rho^2-\delta\lambda^2)(1+\nu)} + \rho^2\right] \quad (16)$$

Equation (16) is the refined first approximation to the radial membrane stress. When $U = 0$ at $\rho = \lambda$, as for cases A and B, A_2 is given by

$$A_2 = -\frac{2}{1-\lambda^2}\int_\lambda^1 \rho G(\rho)\,d\rho \quad (17)$$

and when $S_r = 0$ at $\rho = \lambda$, as for case C, A_2 is given by

$$A_2 = -\frac{2\lambda^2}{[(1-\nu)+(1+\nu)\lambda^2]}\times\left[\frac{(1-\nu)}{\lambda^2}\int_\lambda^1 \rho G(\rho)d\rho + G(\lambda) - \frac{1}{\lambda^2}\left(\frac{A_1-\lambda^2}{1-\delta}\right)^2\right] \quad (18)$$

The refined approximation to the circumferential stress follows directly from the equilibrium conditions.

Tabulation of Coefficients

The analysis now is specialized to the investigation of the problems represented as cases A–C by evaluating A_1, A_2, and α_1, subject to the boundary conditions at $\rho = \lambda$.

Case A: Annulus fixed at both edges

$$\delta = -\left(\frac{1-\nu}{1+\nu}\right); A_1 = \delta\lambda^2 - \frac{1-\lambda^2}{\ln[\lambda^2(1-\delta)/(1-\delta\lambda^2)]} \quad (19a)$$

$$\alpha_1^3 = 1 + \lambda^2(1-4\delta) + 2(A_1-\delta\lambda^2)\times\left[\frac{\delta(A_1-\delta\lambda^2)}{(1-\delta)(1-\delta\lambda^2)} - 1\right] \quad (19b)$$

$$A_2 = -\frac{1}{4}\left[(3-\nu)(1+\lambda^2) + \frac{4\delta(A_1-\delta\lambda^2)^2}{(1-\delta\lambda^2)(1-\delta)} - 2(1+\nu)(A_1+\delta\lambda^2) - 4(1-\nu)\times\left(\frac{\ln(1-\delta\lambda^2) - \lambda^2\ln[\lambda^2(1-\delta)]}{\ln(1-\delta\lambda^2) - \ln[\lambda^2(1-\delta)]}\right)\right] \quad (19c)$$

Case B: Annulus fixed at the outer edge with a rigid plug in the interior

$$\delta = -(1-\nu)/(1+\nu) \qquad A_1 = \lambda^2 \quad (20a)$$

$$\alpha_1^3 = 1 + \lambda^2 - 2\lambda^2(1-\delta)\left[\frac{2-3\delta\lambda^2}{1-\delta\lambda^2}\right] - \frac{2\lambda^4}{1-\lambda^2}(1-3\delta)(1-\delta)\ln\left(\frac{\lambda^2(1-\delta)}{1-\delta\lambda^2}\right) \quad (20b)$$

$$A_2 = -\frac{1}{4}\left\{(3-\nu)(1+\lambda^2) + 4\lambda^2(1-\delta)\times\left[\frac{\delta\lambda^2}{1-\delta\lambda^2} - (1+\nu)\right] - \frac{2\lambda^2(1-\delta)}{1-\lambda^2}\times\left[(1+\nu)(1-3\delta)\lambda^2\ln\left(\frac{\lambda^2(1-\delta)}{1-\delta\lambda^2}\right) + 2(1-\nu)\{\ln(1-\delta\lambda^2) - \lambda^2\ln[\lambda^2(1-\delta)]\}\right]\right\} \quad (20c)$$

Case C: Annulus fixed at the outer edge and free of tractions at the inner edge

$$\delta = 1 \qquad A_1 = \lambda^2 \quad (21a)$$

$$\alpha_1^3 = \frac{(1-\nu)(1-\lambda^4)}{(1+\nu)\lambda^2 + (1-\nu)} \quad (21b)$$

$$A_2 = -\frac{(1-\nu)}{4}\left[\frac{(3-\nu)+(1+\nu)\lambda^4}{(1-\nu)+(1+\nu)\lambda^2}\right] \quad (21c)$$

Uniformly Loaded Solid Circular Membrane, Case D

The problem of a uniformly loaded solid circular membrane supported at the outer edge [Eqs. (5d)] is now examined. The solution for this case can be obtained by taking the limit of the solution of case B, (20), as λ approaches zero.

Thus, the first approximation to the deflection is

$$W_I = \frac{1}{\alpha_1}\left(\frac{1-\nu}{2}\right)^{1/3}(1-\rho^2) \quad (22a)$$

and the radial membrane stress corresponding to the refined first approximation is

$$S_r^{IR} = \frac{1}{4\alpha_1^2}\left(\frac{1-\nu}{2}\right)^{2/3}\left[\frac{3-\nu}{1-\nu} - \rho^2\right] \quad (22b)$$

where $\alpha_1^3 = 1$.

The accuracy of the technique employed is demonstrated by developing a *second set of approximate solutions* for this problem. Proceeding as before, take the form of the radial membrane stress in accordance with (22b), where the magnitude of the stress field is now governed by the characteristic unknown α_2. Consequently,

$$S_r^{II} = \frac{\alpha_2}{4\alpha_1^2}\left(\frac{1-\nu}{2}\right)^{2/3}\left[\frac{3-\nu}{1-\nu} - \rho^2\right] \quad (23)$$

Substituting (23) into (4), integrating, and making use of the conditions $W = 0$ at $\rho = 1$ and $dW/d\rho = 0$ at $\rho = 0$, one finds

$$W_{II} = \frac{\alpha_1^2}{\alpha_2}\left(\frac{2}{1-\nu}\right)^{2/3}\ln\left(\frac{(3-\nu)-(1-\nu)\rho^2}{2}\right) \quad (24)$$

which is the second approximation to the deflection.

Again, the characteristic unknown α_2 is found by solving (3f) for U and then employing the conditions $U = 0$ at $\rho = 1$ and $\rho = 0$. After putting (23) and (24) in (3f), integrating,

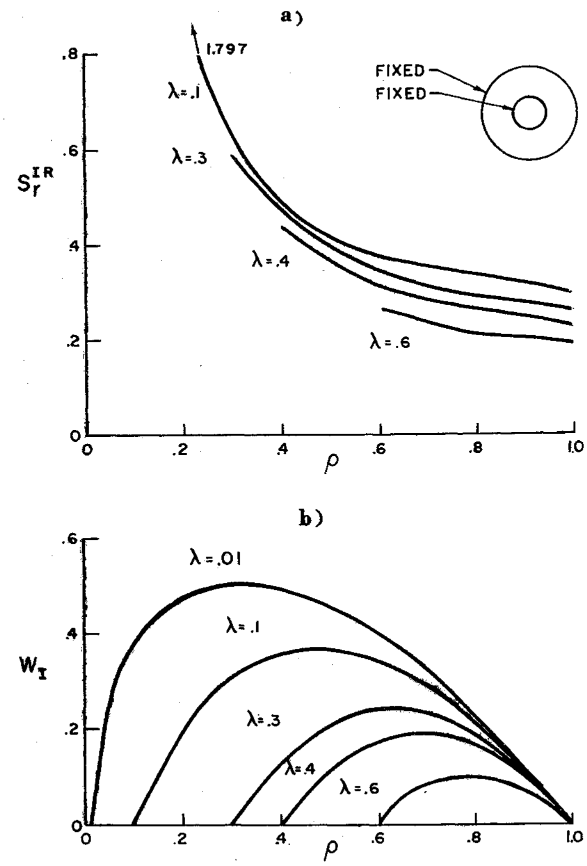

Fig. 1 Case A: Radial stress and deflection for a uniformly loaded annulus fixed at both edges.

and evaluating the constants, one finds

$$\alpha_2^3 = \frac{4}{1-\nu}\left[1 - \frac{2}{1-\nu}\ln\left(\frac{3-\nu}{2}\right)\right] \tag{25}$$

Once again, a refined description of the membrane stresses is obtained by following the same procedure employed previously [Eqs. (14–18)]. Thus, putting (24) into (3a), the radial displacement U_{IIR}, associated with (24) and the equilibrium equation, is found. Integrating the resultant differential equation, and satisfying the boundary conditions at $\rho = 0$ and 1, and making use of the stress displacement relations, one obtains

$$S_r^{IIR} = \frac{\alpha_1}{2\alpha_2^2}\left(\frac{2}{1-\nu}\right)^{4/3}\left\{1 - \left(\frac{1+\nu}{1-\nu}\right)\ln\left(\frac{3-\nu}{2}\right) + \frac{1}{\rho^2}\ln\left[1 - \left(\frac{1-\nu}{3-\nu}\right)\rho^2\right]\right\} \tag{26}$$

Numerical Results

Numerical results for the deflection and the radial membrane stress for cases A–D are shown in Figs. 1–4 for selected values of the radius ratio λ and with Poisson's ratio taken as 0.3.

Examining Figs. 2 and 3, it is seen that the two extreme cases of support at the inner edge (rigid plug and free edge) give rise to stress and deflection patterns inverse to each other. Figure 5 illustrates this situation for $\lambda = 0.01$. In addition, it is noted that the influence of the inner boundary ($\lambda = 0.01$) is confined to a narrow region adjacent to the hole. Elsewhere, the radial membrane stresses cannot be distinguished from those associated with a solid circular membrane.

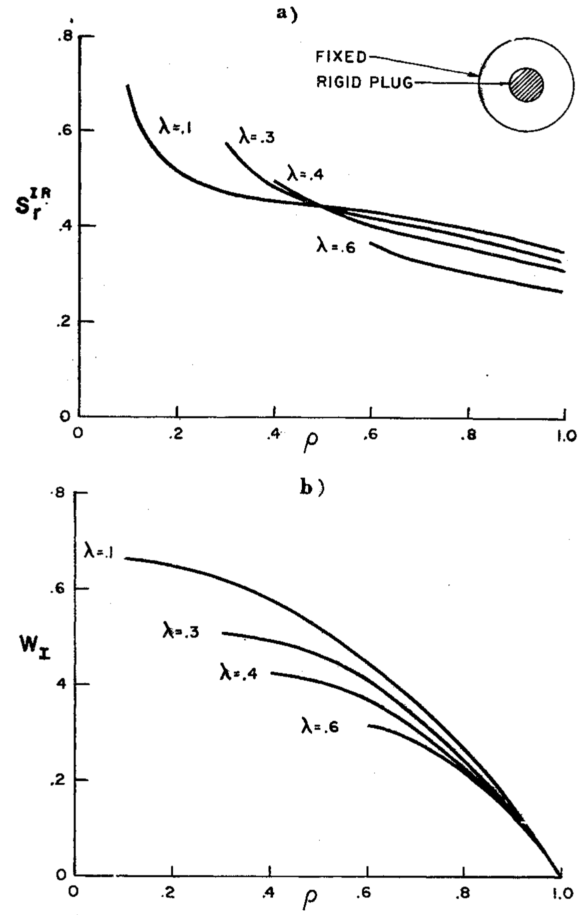

Fig. 2 Case B: Radial stress and deflection for a uniformly loaded annulus fixed at the outer edge with a rigid plug in the interior.

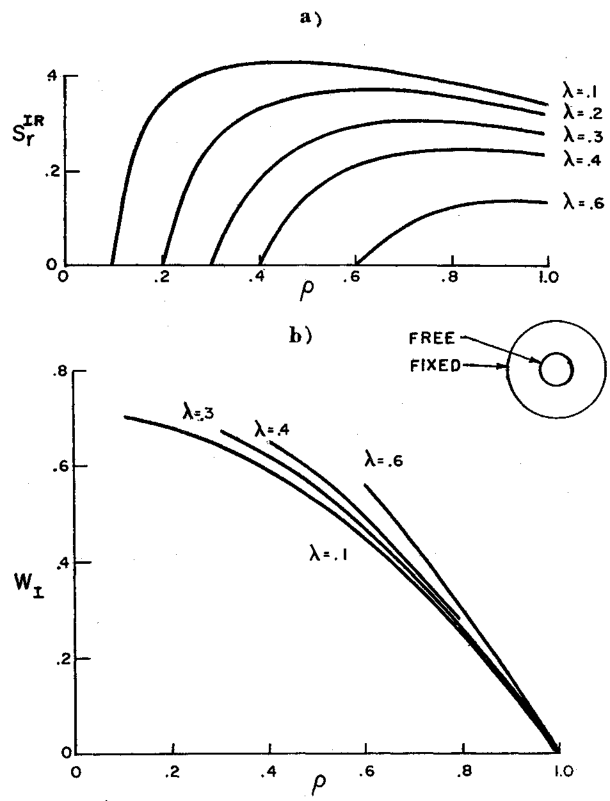

Fig. 3 Case C: Radial stress and deflection for a uniformly loaded annulus fixed at the outer edge and free of tractions at the interior edge.

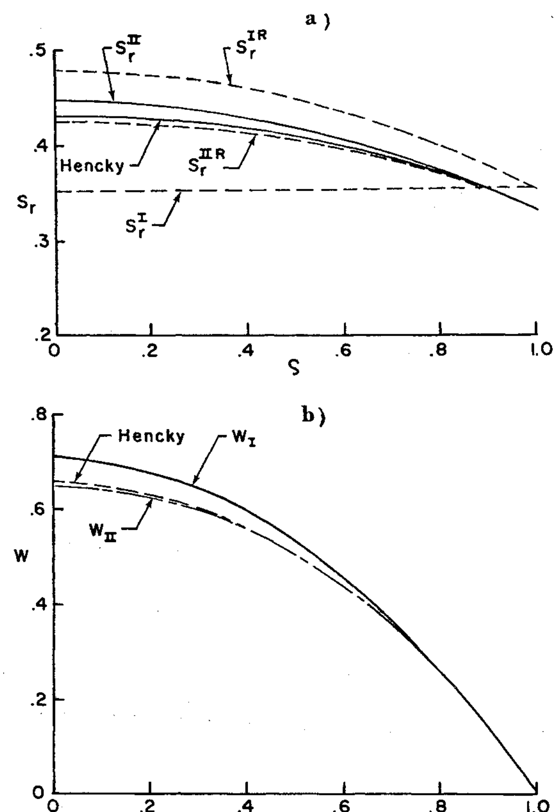

Fig. 4 **Case D: Radial stress and deflection for a uniformly loaded solid membrane fixed at the outer edge.**

The numerical results for the different orders of approximation for solutions to case D are compared with Hencky's[7, 8]‡ more accurate power series solution in Fig. 4.

For $\nu = 0.3$, the first and second approximations to the maximum deflection are, from (22a) and (24),

$$W_{I\ \max} = 0.705 \qquad W_{II\ \max} = 0.648 \tag{27a}$$

compared with Hencky's solution for the circular membrane

$$W_{\max} = 0.6536 \tag{27b}$$

Comparison of the coefficients in (28) shows that the error in the first approximation is $+7.864\%$, and the error in the second approximation is -0.86%, when compared with Hencky's results.

Similarly, the maximum radial stress associated with (22b) and (26) is found:

$$S_r^{IR}{}_{\max} = 0.472 \qquad S_r^{IIR}{}_{\max} = 0.426 \tag{28a}$$

compared with Hencky's solution for the circular membrane

$$S_{r\ \max} = 0.431 \tag{28b}$$

It is anticipated that the first approximate solutions presented for cases A–C will be of the same order of accuracy as that displayed by the first approximate solution for case D, the solid circular membrane.

‡ A numerical error in Hencky's frequently quoted results is pointed out by Chein in Ref. 8.

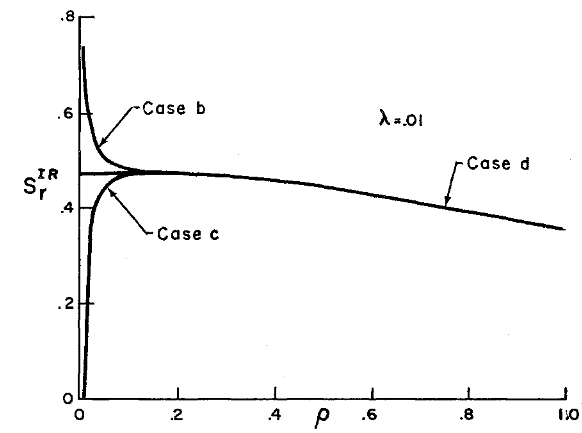

Fig. 5 **Comparison of the radial stresses for cases B, C ($\lambda = 0.01$), and D.**

Conclusions

A technique has been employed to obtain approximate solutions for the large deflection of normally loaded annular membranes. The resultant solutions satisfy the usual equilibrium equations but violate the compatibility of the membrane strains. However, employing this method has significantly reduced the analytical and computational effort necessary to obtain solutions of the four examples presented. The accuracy of the approximate solutions, as can be seen by the comparisons made with Hencky's solution for the solid membrane, appears to be suitable for engineering analyses.

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